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A Constructive Development of Chebyshev Approximation Theory

DOUGLAS S. BRIDGES

University College, Buckingham MK18 1EG, England

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1. INTRODUCTION

Let X be a compact subset of the real line $\mathbb{R}$, $C(X)$ the real algebra of continuous mappings of X into $\mathbb{R}$, and $\| \|_X$ the sup norm on $C(X)$. We shall discuss the characterisation, existence and uniqueness of best approximations to elements of $C(X)$ by linear combinations of functions which form a Chebyshev system. (In most of our work, X will be either an interval or a finite set; but we shall not make any restriction on the compact set X until Section 2, below.)

The classical treatment of these problems is well known and extensively documented (cf. [6–10]). What is distinctive about our discussion is that we work entirely within the framework of constructive mathematics (as developed in [1, 2]). We do so for reasons that we have stated elsewhere [2, 3], and shall not repeat at length here; suffice it to say that a constructive analysis of the sort we shall carry out provides numerical estimates which cannot be obtained by the "existential" techniques of classical mathematics. (Incidentally, this remark is not intended to denigrate classical mathematics: at all stages of our investigation, the classical theory played an indispensable role of guidance and motivation.)

Let n be a positive integer and $\phi_1 ,..., \phi_n$ elements of $C(X)$. We say that $\{\phi_1 ,..., \phi_n\}$ is a *Chebyshev system* (over X) if the following *condition of Haar* is satisfied:

if $K_1 ,..., K_n$ are pairwise disjoint compact subsets of X, then

$$\inf\{|\det[\phi_j(x_i)]|: \forall\, i(x_i \in K_i)\} > 0.$$

(Note that, for constructive purposes, K_i, K_j are *disjoint* if $\inf\{| x_i - x_j |: x_i \in K_i , x_j \in K_j\} > 0$.)

For example, $\{1, x,..., x^{n-1}\}$ is a Chebyshev system over any compact interval in $\mathbb{R}$; and $\{1, \cos x, \sin x, \cos 2x, \sin 2x,..., \cos nx, \sin nx\}$ is a Chebyshev system over any compact subinterval of $[0, 2\pi)$.

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Note that, if $\{\phi_1, ..., \phi_n\}$ is a Chebyshev system over X, $x_1, ..., x_n$ are distinct points of X and $\xi_1, ..., \xi_n$ are real numbers, then there exists a unique linear combination ψ of $\phi_1, ..., \phi_n$ such that $\psi(x_k) = \xi_k$ for $k = 1, ..., n$. It follows from this that the functions $\phi_1, ..., \phi_n$ are linearly independent.

The reader will have observed that our constructive Haar condition is more complicated than its classical counterpart (to which it is equivalent classically). The reason for this is that there is no known constructive proof that a continuous mapping of a compact metric space into the positive real line has a positive lower bound. This state of affairs has a considerable effect on some of our later work. Indeed, while we do not expect that the classical theorem in question will prove to be essentially nonconstructive [2, Chap. 1, Section 5], the complicated analysis required to show that it obtains in special cases of interest leads us to believe that a constructive proof of the theorem is unlikely to be found.

To reinforce these remarks, we end this section of our paper with a proposition of great importance for our subsequent analysis.

1.1. PROPOSITION. *Let $n \geqslant 2$, $\delta > 0$, and suppose that there exist n points $\xi_1, ..., \xi_n$ of X with $\min_{1 \leqslant i<j \leqslant n} |\xi_i - \xi_j| \geqslant \delta$. A necessary and sufficient condition that the elements $\phi_1, ..., \phi_n$ of $C(X)$ form a Chebyshev system is that: for each α in $(0, \delta]$, there exists $\beta > 0$ such that $|\det[\phi_j(x_i)]| \geqslant \beta$ whenever $x_1, ..., x_n$ belong to X and $\min_{1 \leqslant i<j \leqslant n} |x_i - x_j| \geqslant \alpha$.*

Proof. Suppose that $\phi_1, ..., \phi_n$ form a Chebyshev system, and let $0 < \alpha \leqslant \delta$. Let $\{\zeta_1, ..., \zeta_\nu\}$ be an $\alpha/8$-net of X, and construct sets A, B so that

$$\begin{aligned} A \cup B &= \{1, ..., \nu\}^2, \\ (j, k) \in A &\Rightarrow |\zeta_j - \zeta_k| > \alpha/2, \\ (j, k) \in B &\Rightarrow |\zeta_j - \zeta_k| < 3\alpha/4. \end{aligned}$$

Let

$$S = \{i \in \{1, ..., \nu\}^n : \forall j < k\ ((i(j), i(k)) \in A)\}.$$

Let $x_1, ..., x_n$ be points of X such that $\min_{1 \leqslant i<j \leqslant n} |x_i - x_j| \geqslant \alpha$, and choose i in $\{1, ..., \nu\}^n$ so that $|x_j - \zeta_{i(j)}| < \alpha/8$ for $j = 1, ..., n$. Then for $j < k$ we have

$$\begin{aligned} |\zeta_{i(j)} - \zeta_{i(k)}| &\geqslant |x_j - x_k| - |x_j - \zeta_{i(j)}| - |x_k - \zeta_{i(k)}| \\ &> 3\alpha/4; \end{aligned}$$

so that $(i(j), i(k)) \notin B$, and therefore $(i(j), i(k)) \in A$. Thus $i \in S$, and S is nonempty. For each s in S, choose $\beta(s) > 0$ so that $|\det[\phi_j(\xi_k)]| \geqslant \beta(s)$ whenever $|\xi_k - \zeta_{s(k)}| \leqslant \alpha/8$ for $k = 1, ..., n$. (To do this, first choose r in $(\alpha/8, \alpha/4)$ so that the closed ball $\overline{B}(\zeta_{s(j)}, r)$ of centre $\zeta_{s(j)}$ and radius r is compact for each j. As $|\zeta_{s(j)} - \zeta_{s(k)}| > \alpha/2$ for $j \neq k$, the balls $\bar{B}(\zeta_{s(j)}, r)$ are pairwise disjoint, and the Haar condition can be applied to them to produce the number $\beta(s)$.) With $\beta = \min\{\beta(s): s \in S\}$, we now have $|\det[\phi_j(x_k)]| \geqslant \beta(i) \geqslant \beta$. This completes the proof of necessity of the condition stated in our proposition; the proof of sufficiency is routine, and will be omitted. ∎

By evaluating the appropriate Vandermonde determinant, it is easy to show that, for the Chebyshev system $\{1, x,..., x^{n-1}\}$ over a compact interval, we can take $\beta = \alpha^{n(n-1)/2}$ in the above proposition.

2. Basic Properties of Chebyshev Systems over [0, 1]

For the rest of this paper, n will be a fixed positive integer, $\{\phi_1,..., \phi_n\}$ a Chebyshev system over X, H the n-dimensional real linear subspace of $C(X)$ spanned by $\{\phi_1,..., \phi_n\}$, and a an element of $C(X)$. We want to compute and characterise a *best (Chebyshev) approximant* of a in H: that is, an element b of H such that

$$\| a - b \|_X = \operatorname{dist}(a, H) = \inf\{\| a - \psi \|_X: \psi \in H\}.$$

Note that $\operatorname{dist}(a, H)$ is computable, by [3, 2.1].

It will be helpful to introduce some notational shorthand at this point. Given $\mathbf{a} = (a_1,..., a_m)$ in $\mathbb{R}^m$, we write $\| \mathbf{a} \|_2$ for $(\sum_{j=1}^m a_j^2)^{1/2}$. We also write

$$\boldsymbol{\phi}(x) = (\phi_1(x),..., \phi_n(x)) \qquad (x \in X)$$

and

$$\| \boldsymbol{\phi} \| = \sup\{\| \boldsymbol{\phi}(x)\|_2: x \in X\}$$

(the latter being computable, by [2, Chap. 2, 4.4]). If $n \geqslant 2$ and α, β are as in 1.1, we write $\beta(\alpha)$ for β; if $n = 1$ and $\alpha > 0$, we write $\beta(\alpha)$ for $\inf\{|\phi_1(x)|: x \in X\}$; in either case, we then define

$$\gamma(\alpha) = \min(\| \boldsymbol{\phi} \|, \beta(\alpha)/n^{1/2}(n-1)! \prod_{i=1}^{n} (1 + \| \phi_i \|_X)).$$

(The notations $\beta(\alpha)$, $\gamma(\alpha)$ represent convenient, but dispensable, applications of the Axiom of Choice.)

By far the most interesting Chebyshev approximation problems occur when X is a compact interval in $\mathbb{R}$ (cf. [3, Sections 4–6]). To deal with this case, we shall assume from here until Section 5, below, that $X = [0, 1]$, and we shall write $\| \ \|$ for $\| \ \|_{[0,1]}$. Note that, if $(x_1,..., x_{n+1})$ is a strictly increasing sequence of $n+1$ points of $[0, 1]$ and $\lambda_1, \ldots, \lambda_{n+1}$ are real numbers with $\sum_{i=1}^{n+1} |\lambda_i| = 1$, then (by the Cauchy–Schwarz inequality in $\mathbb{R}^n$)

$$\begin{aligned}
\left\| \sum_{j=1}^{n} a_j \sum_{i=1}^{n+1} \lambda_i \phi_j(x_i) \right\| &\leqslant \| \mathbf{a} \|_2 \left\| \sum_{i=1}^{n+1} \lambda_i \boldsymbol{\phi}(x_i) \right\|_2 \\
&\leqslant \| \mathbf{a} \|_2 \sum_{i=1}^{n+1} |\lambda_i| \, \| \boldsymbol{\phi}(x_i) \|_2 \\
&\leqslant \| \mathbf{a} \|_2 \sum_{i=1}^{n+1} |\lambda_i| \, \| \boldsymbol{\phi} \| \\
&= \| \mathbf{a} \|_2 \, \| \boldsymbol{\phi} \|.
\end{aligned}$$

Our first task, carried out in parts 2.1–2.4, is to obtain some basic numerical and interpolatory properties of our Chebyshev system.

2.1. Lemma. *Let $K_1, \ldots, K_n$ be pairwise disjoint compact subsets of* $[0, 1]$, $\delta = \inf\{|\det[\phi_j(x_i)]| : \forall i (x_i \in K_i)\}$, *and* $\psi = \sum_{j=1}^{n} a_j \phi_j$. *Then*

$$\max_{i=1,\ldots,n} \inf\{|\psi(x)| : x \in K_i\} \geqslant \delta \| \mathbf{a} \|_2 / n^{3/2} (n-1)! \prod_{r=1}^{n} (1 + \| \phi_r \|).$$

Proof. With

$$\mu = \delta / n^{3/2} (n-1)! \prod_{r=1}^{n} (1 + \| \phi_r \|),$$

$$M = \max_{i=1,\ldots,n} \inf\{|\psi(x)| : x \in K_i\},$$

suppose that $M < \mu \| \mathbf{a} \|_2$. For each $i \in \{1, \ldots, n\}$, choose x_i in K_i so that $|\psi(x_i)| < \mu \| \mathbf{a} \|_2$. Let Φ_{rs} be the cofactor of $\phi_s(x_r)$ in the n-by-n matrix $[\phi_j(x_i)]$, $\Delta = \det[\phi_j(x_i)]$, and note that

$$|\Phi_{ij}| \leqslant (n-1)! \prod_{r=1, r \neq j}^{n} \| \phi_r \| \leqslant (n-1)! \prod_{r=1}^{n} (1 + \| \phi_r \|).$$

Using Cramer's rule, we obtain

$$\begin{aligned}
|a_j| &= \left| \Delta^{-1} \sum_{i=1}^{n} \psi(x_i) \Phi_{ij} \right| \\
&\leqslant \delta^{-1} \sum_{i=1}^{n} |\psi(x_i)| \, |\Phi_{ij}| \\
&< \delta^{-1} \sum_{i=1}^{n} \mu \| \mathbf{a} \|_2 (n-1)! \prod_{r=1}^{n} (1 + \| \phi_r \|) \\
&= \| \mathbf{a} \|_2 / n^{1/2}.
\end{aligned}$$

This leads to the contradiction

$$\| \mathbf{a} \|_2 = \left(\sum_{j=1}^{n} a_j^2 \right)^{1/2} < \left(\sum_{j=1}^{n} \| \mathbf{a} \|_2^2/n \right)^{1/2} = \| \mathbf{a} \|_2 .$$

Thus, in fact, $M \geqslant \mu \| \mathbf{a} \|_2$. ∎

2.2. *Remark.* In the notation of Lemma 2.1, suppose that $n \geqslant 2$ and that, for each $i \in \{2,..., n\}$, there exists x_i in K_i with $\psi(x_i) = 0$. Then, for each x_1 in K_1 and each j in $\{1,..., n\}$ we have $a_j = \Delta^{-1}\psi(x_1)\psi_{1j}$; whence

$$\| \mathbf{a} \|_2^2 \leqslant \Delta^{-2}\psi(x_1)^2 \sum_{j=1}^{n} \Phi_{1j}^2$$

$$\leqslant \delta^{-2}\psi(x_1)^2\, n((n-1)! \prod_{r=1}^{n} (1 + \| \phi_r \|))^2.$$

This gives

$$\inf\{| \psi(x)| : x \in K_1\} \geqslant \delta \| \mathbf{a} \|_2 / n^{1/2}(n-1)! \prod_{r=1}^{n} (1 + \| \phi_r \|),$$

a strengthening of the estimate in Lemma 2.1.

For each of the next three results, the reader is invited to provide himself with the modifications of their classical proofs which will yield constructive ones.

2.3. Lemma. *Let $n \geqslant 2$, $\psi \in H$, $\| \psi \| > 0$, and suppose that ψ has zeroes at $n-1$ distinct points of* $[0, 1]$. *Then $\psi(x)$ changes sign at each zero of ψ in* $(0, 1)$.

2.4. Lemma. *Let $n \geqslant 3$, and let $x_1, ..., x_{n-2}$ be $n-2$ distinct points of* $(0, 1)$. *Then there exists ψ in H such that*

(a) *for each i in $\{1,..., n-2\}$, $\psi(x_i) = 0$ and $\psi(x)$ changes sign at x_i;*

(b) *for each compact $K \subset [0, 1]$ which is disjoint from $\{x_i : i = 1,..., n-2\}$,* $\inf\{| \psi(x)| : x \in K\} > 0$.

2.5. Caratheodory's Lemma. *Let m, ν be positive integers with $m > \nu + 1$, $A \subset \mathbb{R}^\nu$ and $\mathbf{x}$ a convex linear combination of m elements of A. Then, for each $\epsilon > 0$, there exists a convex linear combination $\mathbf{y}$ of $m-1$ elements of A such that* $\| \mathbf{x} - \mathbf{y} \| < \epsilon$ [6, p. 17].

We next remark that, $\boldsymbol{\phi}$: $[0, 1] \to \mathbb{R}^n$ being uniformly continuous, $\{\boldsymbol{\phi}(x): x \in [0, 1]\}$ is totally bounded, as its convex hull. In particular, this ensures that

$$\begin{aligned}\operatorname{dist}(\mathbf{0}, \operatorname{co}\{\boldsymbol{\phi}(x): x \in [0, 1]\}) \\ = \inf\{\|\boldsymbol{\xi}\|_2: \boldsymbol{\xi} \in \operatorname{co}\{\boldsymbol{\phi}(x): x \in [0, 1]\}\}\end{aligned}$$

is computable. In order to construct a nonvanishing element of H, we show that this distance is positive.

2.6. Lemma. $\operatorname{dist}(\mathbf{0}, \operatorname{co}\{\boldsymbol{\phi}(x): x \in [0, 1]\}) > 0$.

Proof. The proof is in several stages.

2.6.1. *Let* $0 < \alpha < n^{-1}$, *and let* $x_1, \ldots, x_{n+1}$ *be points of* $[0, 1]$ *such that* $\min_{k=1,\ldots,n}(x_{k+1} - x_k) \geqslant \alpha$. *Let* $\rho_1, \ldots, \rho_{n+1}$ *be nonnegative numbers with* $\sum_{i=1}^{n+1} \rho_i = 1$. *Then* $\|\sum_{i=1}^{n+1} \rho_i \boldsymbol{\phi}(x_i)\|_2 \geqslant n^{-1}\gamma(\alpha)$.

Let $r \in \{1, \ldots, n\}$, and construct $\psi = \sum_{j=1}^{n} a_j \phi_j$ in H so that

$$\begin{aligned}\psi(x_i) &= 1 \quad \text{if} \quad i = r, \\ &= 0 \quad \text{if} \quad i \in \{1, \ldots, n+1\}, i \neq r \quad \text{and} \quad i \neq r+1.\end{aligned}$$

Applying 2.2 to the disjoint compact sets $[x_r, x_{r+1}]$, $\{x_i\}$ $(i \in \{1, \ldots, n+1\}$, $i \neq r$, $i \neq r+1)$, we have

$$\inf\{|\psi(x)|: x \in [x_r, x_{r+1}]\} \geqslant \gamma(\alpha) \|\mathbf{a}\|_2 > 0.$$

As $\psi(x_r) > 0$, [2, Chap. 2, 3.3] ensures that $\psi(x_{r+1}) \geqslant \gamma(\alpha) \|\mathbf{a}\|_2$; whence

$$\begin{aligned}(\rho_r + \rho_{r+1})\, \gamma(\alpha) \|\mathbf{a}\|_2 &\leqslant \rho_r \psi(x_r) + \rho_{r+1}\psi(x_{r+1}) \\ &= \sum_{i=1}^{n+1} \rho_i \psi(x_i) \\ &= \sum_{j=1}^{n} a_j \sum_{i=1}^{n+1} \rho_i \phi_j(x_i) \\ &\leqslant \|\mathbf{a}\|_2 \left\| \sum_{i=1}^{n+1} \rho_i \boldsymbol{\phi}(x_i) \right\|_2.\end{aligned}$$

Thus

$$0 \leqslant (\rho_j + \rho_{j+1})\, \gamma(\alpha) \leqslant \left\| \sum_{i=1}^{n+1} \rho_i \boldsymbol{\phi}(x_i) \right\|_2 \qquad (j = 1, \ldots, n)$$

and so

$$
\begin{aligned}
0 < \gamma(\alpha) &= \sum_{i=1}^{n+1} \rho_i \gamma(\alpha) \\
&\leqslant \left(\sum_{i=1}^{n+1} 2\rho_i - (\rho_1 + \rho_{n+1}) \right) \gamma(\alpha) \\
&= \sum_{i=1}^{n} (\rho_i + \rho_{i+1})\, \gamma(\alpha) \\
&\leqslant n \left\| \sum_{i=1}^{n+1} \rho_i \phi(x_i) \right\|_2 .
\end{aligned}
$$

Division by n completes the proof of 2.6.1.

2.6.2.

$$\inf\{\|\phi(x)\|_2 : x \in [0, 1]\} > 0.$$

Given x in $[0, 1]$, compute $x_1, \ldots, x_{n+1}$ in $[0, 1]$ and r in $\{1, \ldots, n+1\}$ so that $x_r = x$ and $\min_{k=1,\ldots,n}(x_{k+1} - x_k) \geqslant 1/4n$. Then, with $\rho_r = 1$, $\rho_i = 0$ for $i \neq r$, part 2.6.1 entails

$$\|\phi(x)\|_2 = \left\| \sum_{i=1}^{n+1} \rho_i \phi(x_i) \right\|_2 \geqslant n^{-1}\gamma(1/4n) > 0.$$

2.6.3. *For each* m *in* $\{1, \ldots, n+1\}$, *there exists* $c_m > 0$ *such that:* $\| \sum_{i=1}^{m} \rho_i \phi(x_i) \|_2 \geqslant c_m$ *whenever* $\rho_1, \ldots, \rho_m$ *are nonnegative numbers,* $\sum_{i=1}^{m} \rho_i = 1$ *and* $x_1, \ldots, x_m$ *are distinct points of* $[0, 1]$.

The case $m = 1$ is just case 2.6.2. Let $k \in \{1, \ldots, n\}$, suppose we have proved 2.6.3 for $m = k$, and consider the case $m = k + 1$. Let δ be a modulus of uniform continuity for the mapping $F: (\boldsymbol{\rho}, \mathbf{x}) \to \| \sum_{i=1}^{n+1} \rho_i \phi(x_i) \|_2$ on the compact subset

$$\left\{ \boldsymbol{\rho} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \rho_i = 1, \forall i(\rho_i \geqslant 0) \right\} \times [0, 1]^{n+1}$$

of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, where the latter is taken with norm

$$\|(\boldsymbol{\rho}, \mathbf{x})\| = \max_{i=1,\ldots,n+1} \max(|\rho_i|, |x_i|).$$

Let

$$
\begin{aligned}
\alpha &= \min(2^{-1}\delta(\tfrac{1}{2} c_k), 1/2n(n+1)), \\
c_m &= \min(\tfrac{1}{2} c_k, n^{-1}\gamma(\alpha)).
\end{aligned}
$$

Let $\rho_1, \ldots, \rho_{k+1}$ be nonnegative numbers with $\sum_{i=1}^{k+1} \rho_i = 1$, $0 \leqslant x_1 < x_2 < \cdots < x_{k+1} \leqslant 1$ and $\mu = \min_{i=1,\ldots,k}(x_{i+1} - x_i)$. We have either $\alpha < \mu$ or $\mu < 2\alpha$. In the former case, if $k < n$ we set $\rho_i = 0$ for $i = k + 2, \ldots,$

$n+1$, and choose $x_{k+2},\ldots, x_{n+1}$ in $[0, 1]$ so that $\min_{1\leqslant i<j\leqslant n+1} | x_i - x_j | > \alpha$. This latter choice is possible as

$$\max(x_1, \max_{i=1,\ldots,k}(x_{i+1} - x_i), 1 - x_{k+1}) \\ \geqslant 1/(k+2) \geqslant 1/(n+1) > n\alpha.$$

By part 2.6.1, we have

$$\left\| \sum_{i=1}^{k+1} \rho_i \phi(x_i) \right\|_2 = \left\| \sum_{i=1}^{n+1} \rho_i \phi(x_i) \right\|_2 \geqslant n^{-1}\gamma(\alpha) \geqslant c_m .$$

In the case $\mu < 2\alpha$, choosing r in $\{1,\ldots, k\}$ with $x_{r+1} - x_r < 2\alpha$, we define

$$\begin{aligned} \xi_i &= x_i && \text{if } i \in \{1,\ldots,k+1\}, i \neq r+1, \\ &= x_r && \text{if } i = r+1, \\ \rho_i' &= \rho_i && \text{if } i \in \{1,\ldots,k+1\}, i \neq r \text{ and } i \neq r+1, \\ &= \rho_r + \rho_{r+1} && \text{if } i = r. \end{aligned}$$

If $k < n$, we also set $\rho_i = 0$, $\xi_i = x_i = 1$ for $i = k+2,\ldots, n+1$. Then each ρ_i' is nonnegative,

$$\sum_{i=1, i\neq r+1}^{k+1} \rho_i' = \sum_{i=1}^{n+1} \rho_i = 1$$

and

$$\begin{aligned} \sum_{i=1}^{n+1} \rho_i \phi(\xi_i) &= \sum_{i=1, i\neq r, i\neq r+1}^{k+1} \rho_i \phi(x_i) + (\rho_r + \rho_{r+1})\, \phi(x_r) \\ &= \sum_{i=1, i\neq r+1}^{k+1} \rho_i' \phi(x_i). \end{aligned}$$

As also

$$\|(\boldsymbol{\rho}, \boldsymbol{\xi}) - (\boldsymbol{\rho}, \mathbf{x})\| = x_{r+1} - x_r < 2\alpha \leqslant \delta(\tfrac{1}{2} c_k),$$

we now have

$$\begin{aligned} \left\| \sum_{i=1}^{m} \rho_i \phi(x_i) \right\|_2 &= \left\| \sum_{i=1}^{n+1} \rho_i \phi(x_i) \right\|_2 \\ &\geqslant \left\| \sum_{i=1}^{n+1} \rho_i \phi(\xi_i) \right\|_2 - \tfrac{1}{2} c_k \\ &= \left\| \sum_{i=1, i\neq r+1}^{k+1} \rho_i' \phi(x_i) \right\|_2 - \tfrac{1}{2} c_k \\ &\geqslant \tfrac{1}{2} c_k \\ &\geqslant c_m . \end{aligned}$$

This completes the proof of part 2.6.3.

With c_{n+1} as in part 2.6.3, we now suppose that

$$\operatorname{dist}(\mathbf{0}, \operatorname{co}\{\boldsymbol{\phi}(x): x \in [0, 1]\}) < c_{n+1}$$

and compute nonnegative numbers $\rho_1, ..., \rho_m$, and points $x_1, ..., x_m$ of $[0, 1]$, such that $\sum_{i=1}^{m} \rho_i = 1$ and $\| \sum_{i=1}^{m} \rho_i \boldsymbol{\phi}(x_i)\|_2 < c_{n+1}$. Repeated application of Lemma 2.5, if necessary, allows us to take $m \leqslant n + 1$; from which it easily follows that we can assume that $m = n + 1$. As the mapping F (introduced in the proof of part 2.6.3) is uniformly continuous over its domain, we can find nonnegative numbers $\rho_1', ..., \rho_{n+1}'$ and distinct points $x_1', ..., x_{n+1}'$ of $[0, 1]$ such that $\sum_{i=1}^{n+1} \rho_i' = 1$ and $\| \sum_{i=1}^{n+1} \rho_i' \boldsymbol{\phi}(x_i')\|_2 < c_{n+1}$. This contradicts 2.6.3; whence, in fact,

$$\operatorname{dist}(\mathbf{0}, \operatorname{co}\{\boldsymbol{\phi}(x): x \in [0, 1]\}) \geqslant c_{n+1} > 0. \quad \blacksquare$$

2.7. PROPOSITION. *There exists* ψ *in* H *with* $\inf\{\psi(x): x \in [0, 1]\} > 0$

Proof. Choosing c with $0 < c < \operatorname{dist}(\mathbf{0}, \operatorname{co}\{\boldsymbol{\phi}(x): x \in [0, 1]\})$, we apply the Separation Theorem [2, Chap. 3, 3.3] to the subsets $\{\mathbf{0}\}$, $\operatorname{co}\{\boldsymbol{\phi}(x): x \in [0, 1]\}$ of $\mathbb{R}^n$, to construct a normable linear functional u on $\mathbb{R}^n$ such that $u(\boldsymbol{\xi}) > \frac{1}{2}c$ for each $\boldsymbol{\xi}$ in $\operatorname{co}\{\boldsymbol{\phi}(x): x \in [0, 1]\}$. As u is normable, there exists $\mathbf{a}$ in $\mathbb{R}^n$ such that $u(\mathbf{x}) = \sum_{j=1}^{n} a_j x_j$ for each $\mathbf{x}$ in $\mathbb{R}^n$. With $\psi = \sum_{j=1}^{n} a_j \phi_j \in H$, we now have

$$\psi(x) = u(\boldsymbol{\phi}(x)) \geqslant \tfrac{1}{2}c > 0 \qquad (x \in [0, 1]),$$

whence $\inf\{\psi(x): x \in [0, 1]\} > 0$. $\blacksquare$

3. CHARACTERISATION OF BEST CHEBYSHEV APPROXIMANTS

Our next task is to extract the constructive essence from Borel's classical characterisation of best Chebyshev approximants [10, Theorem 3–1]. In order to accomplish this, we require some more definitions and lemmas.

The first of these definitions introduces a constructive substitute for the classical notion of "alternant." Let $p \in H$ and $\epsilon > 0$. By an *ϵ-alternant* of a and p we mean an ordered pair comprising an integer $j \in \{0, 1\}$ and a strictly increasing sequence $(x_1, ..., x_{n+1})$ of $n + 1$ points of $[0, 1]$ such that

$$(-1)^{k-j}(a - p)(x_k) > \| a - p \| - \epsilon \qquad (k = 1, ..., n + 1).$$

If also $0 < \epsilon < \| a - p \|$ and $m \in \{0, ..., n - 1\}$, we define an *(m, ϵ)-prealternant of a and p* to be an ordered pair comprising an integer $j \in \{0, 1\}$ and a strictly increasing sequence $(x_1, ..., x_{2m+4})$ of $2m + 4$ points of $[0, 1]$ such that $x_1 = 0$, $x_{2m+4} = 1$,

$$(-1)^{j}(a - p)(x_2) > \| a - p \| - \epsilon,$$
$$(-1)^{m+1-j}(a - p)(x_{2m+3}) > \| a - p \| - \epsilon,$$

$$(-1)^{k-j}(a-p)(x_r) > \|a-p\| - \epsilon \quad (r = 2k+1, 2k+2;\ k = 1,\dots, m)$$

and

$$\sup\{|(a-p)(x)| : x_{2k} \leqslant x \leqslant x_{2k+1}\} < \|a-p\| \qquad (k = 1,\dots, m+1).$$

3.1. Lemma. *Let $p \in H$, and suppose that $0 < \epsilon < \|a-p\|$. Then either $\|a-p\| > \operatorname{dist}(a, H)$ or there exists a $(0, \epsilon)$-prealternant of a and p.*

Proof. Let m, M be respectively the inf, sup of $a-p$ over $[0, 1]$. Either $\|a-p\| > \min(-m, M)$ or $\min(-m, M) > \|a-p\| - \epsilon$. In the former case, we choose α so that

$$0 < \alpha < 2^{-1}(\|a-p\| - \min(-m, M))$$

and $\psi \in H$ with $0 < \inf\{\psi(x): x \in [0, 1]\}$ and $\|\psi\| \leqslant 1$ (Proposition 2.7). If $\|a-p\| > -m$ (when $\|a-p\| = M$), we set $q = p + \alpha\psi$; so that $q \in H$ and, for each x in $[0, 1]$,

$$
\begin{aligned}
(a-q)(x) &\leqslant \|a-p\| - \alpha \inf \psi,\\
(q-a)(x) &\leqslant \alpha\psi(x) + \sup\{(p-a)(\xi): \xi \in [0, 1]\}\\
&\leqslant \alpha - m\\
&< \|a-p\| - \alpha\\
&\leqslant \|a-p\| - \alpha \inf \psi.
\end{aligned}
$$

Hence

$$\|a-p\| \geqslant \|a-q\| + \alpha \inf \psi > \operatorname{dist}(a, H).$$

We obtain the same inequality in the case $\|a-p\| > M$ by taking $q = p - \alpha\psi$.

On the other hand, if $\min(-m, M) > \|a-p\| - \epsilon$, we argue as in the corresponding part of the proof of [3, 4.1], to show that there exists a $(0, \epsilon)$-prealternant of a and p. ∎

3.2 Lemma. *Let m be an integer, $0 \leqslant m \leqslant n-2$, and $p \in H$. Suppose that $0 < \epsilon < \|a-p\|$, and that there exists an (m, ϵ)-prealternant of a and p. Then either $\|a-p\| > \operatorname{dist}(a, H)$ or there exists an $(m+1, \epsilon)$-prealternant of a and p.*

Proof. Let $(j, (t_1, \dots, t_{2m+4}))$ be an (m, ϵ)-prealternant of a and p, and define

$$\mu = \max_{k=1,\dots,m+2} \sup\{(-1)^{k-j}(a-p)(x): t_{2k-1} \leqslant x \leqslant t_{2k}\}.$$

Either $\| a - p \| > \mu$ or $\mu > \| a - p \| - \epsilon$. In the former case, choose $\alpha > 0$ so that

$$\begin{aligned} &\max(\mu, \max_{k=1,\dots,m+1} \sup\{|(a-p)(x)| : t_{2k} \leqslant x \leqslant t_{2k+1}\}) \\ &\qquad < \| a - p \| - 2\alpha, \end{aligned}$$

let

$$z_k = (t_{2k} + t_{2k+1})/2 \qquad (k = 1,\dots, m + 1)$$

and define

$$\begin{aligned} r &= n - 1 \qquad \text{if} \quad n - m \text{ is even,} \\ &= n - 2 \qquad \text{if} \quad n - m \text{ is odd.} \end{aligned}$$

If $m + 2 < r$, also choose a strictly increasing sequence $(z_{m+2},\dots, z_r)$ of $r - m - 1$ points of (z_1, t_3). Construct ψ in H so that $(-1)^j\psi(t_2) > 0$, $0 < \| \psi \| < \min(\alpha, \| a - p \|/2)$, ψ has its zeroes, and changes sign, at each of $z_1,\dots, z_r$, and

$$0 < \sigma = 2^{-1} \inf\left\{ |\psi(x)| : x \in \bigcup_{i=1}^{m+2} [t_{2i-1}, t_{2i}]\right\}.$$

(This construction is possible in view of 2.2, 2.3 and 2.4.) Let $q = p + \psi$, suppose that $\| a - q \| > \| a - p \| - \sigma$, and choose ζ in $[0, 1]$ so that $|(a - q)(\zeta)| > \| a - p \| - \sigma$. Then

$$\begin{aligned} |(a-p)(\zeta)| &\geqslant |(a-q)(\zeta)| - |(p-q)(\zeta)| \\ &\geqslant \| a - p \| - \sigma - \| \psi \| \\ &\geqslant \| a - p \| - 2\| \psi \| \\ &> \| a - p \| - 2\alpha \\ &> \max_{k=1,\dots,m+1} \sup\{|(a-p)(x)| : t_{2k} \leqslant x \leqslant t_{2k+1}\}. \end{aligned}$$

It follows from this and the uniform continuity of $a - p$ on $[0, 1]$ that there exists i in $\{1,\dots, m + 2\}$ with $t_{2i-1} \leqslant \zeta \leqslant t_{2i}$.

Noting that the zeroes of ψ occur precisely at the points $z_1,\dots, z_r$, and that each interval $[t_{2k}, t_{2k+1}]$ contains an odd number of these zeroes, we have

$$(-1)^{k-j+1}\psi(x) > 0 \qquad (x \in [t_{2k-1}, t_{2k}], \qquad k = 1,\dots, m + 2).$$

Hence

$$\begin{aligned} (-1)^{i-j}(q - a)(\zeta) &= (-1)^{i-j}\psi(\zeta) + (-1)^{i-j}(p - a)(\zeta) \\ &< -\sigma + \| a - p \|. \end{aligned}$$

It follows from this and our choice of ζ that $(-1)^{i-j}(a-q)(\zeta) > \| a-p \| - \sigma$; whence

$$\begin{aligned}(-1)^{i-j}(a-p)(\zeta) &= (-1)^{i-j}(a-q)(\zeta) + (-1)^{i-j}\psi(\zeta)\\ &> \| a-p \| - \sigma - \alpha\\ &> \| a-p \| - 2\alpha\\ &> \mu.\end{aligned}$$

This contradicts the definition of μ; so that

$$\| a-p \| \geqslant \| a-q \| + \sigma > \operatorname{dist}(a, H).$$

On the other hand, if $\mu > \| a-p \| - \epsilon$, we can argue as in the corresponding part of the proof of [3, 4.2], to show that there exists an $(m+1, \epsilon)$-prealternant of a and p. ∎

Our next two results, taken together, form a constructive analogue of Borel's characterisation theorem. We omit the proofs, as they differ in at most trivial details from those of the corresponding theorems in the case of minimax polynomial approximation [3, 4.3 and 4.4].

3.3 PROPOSITION. *Let $p \in H$ and $0 < \epsilon < \| a-p \|$. Then either $\| a-p \| >$ $\operatorname{dist}(a, H)$ or there exists an ϵ-alternant of a and p.*

3.4 THEOREM. *A necessary and sufficient condition that $b \in H$ be a best Chebyshev approximant of a in H is that, for each $\epsilon > 0$, there exists an ϵ-alternant of a and b.*

4. EXISTENCE AND UNIQUENESS OF BEST CHEBYSHEV APPROXIMANTS

Each of the three remaining steps on the path to the construction of best Chebyshev approximants is a stronger, and much more informative, version of a classical counterpart (cf. [8, 3.5.1; 9, Theorem 24]).

4.1 LEMMA. *Let $0 < \alpha \leqslant n^{-1}$, and let $x_1, \ldots, x_{n+1}$ be points of $[0, 1]$ with $\min_{k=1,\ldots,n}(x_{k+1} - x_k) \geqslant \alpha$. Let $\lambda_1 = 1, \lambda_2, \ldots, \lambda_{n+1}$ be real numbers such that $\sum_{i=1}^{n+1} \lambda_i \boldsymbol{\phi}(x_i) = \mathbf{0}$. Then*

$$(\gamma(\alpha)/\| \boldsymbol{\phi} \|)^{r-1} \leqslant (-1)^{r-1}\lambda_r \leqslant (\| \boldsymbol{\phi} \|/\gamma(\alpha))^{r-1} \qquad (*)$$

for each r in $\{1, \ldots, n+1\}$.

Proof. We proceed by induction on r. If $r = 1$, the result is trivial. Let $r \in \{1,..., n\}$, suppose that (*) obtains, and construct $\psi = \sum_{j=1}^{n} a_j\phi_j$ in H so that

$$\begin{aligned} \psi(x_i) &= 1 \qquad \text{if} \quad i = r, \\ &= 0 \qquad \text{if} \quad i \in \{1,..., n+1\}, i \neq r \qquad \text{and} \qquad i \neq r+1. \end{aligned}$$

Applying Remark 2.2 and [2, Chap. 2, 3.3] to the sets $[x_r, x_{r+1}]$, $\{x_i\}$ $(i \in \{1,..., n+1\}, i \neq r, i \neq r+1)$, we see that

$$\inf\{\psi(x)\colon x_r \leqslant x \leqslant x_{r+1}\} \geqslant \gamma(\alpha) \| \mathbf{a} \|_2 > 0.$$

On the other hand,

$$\begin{aligned} \lambda_r\psi(x_r) + \lambda_{r+1}\psi(x_{r+1}) &= \sum_{i=1}^{n+1} \lambda_i\psi(x_i) \\ &= \sum_{j=1}^{n} a_j \sum_{i=1}^{n+1} \lambda_i\phi_j(x_i) \\ &= 0, \end{aligned}$$

whence $(-1)^r\lambda_{r+1} = (-1)^{r-1}\lambda_r\psi(x_r)/\psi(x_{r+1})$. Recalling that $|\psi(x)| \leqslant \| \mathbf{a} \|_2 \times \| \boldsymbol{\phi} \|$ for each x in $[0, 1]$, we now obtain

$$(-1)^r\lambda_{r+1} \geqslant (\gamma(\alpha)/\| \boldsymbol{\phi} \|)^{r-1}\gamma(\alpha) \| \mathbf{a} \|_2/\| \mathbf{a} \|_2 \| \boldsymbol{\phi} \| = (\gamma(\alpha)/\| \boldsymbol{\phi} \|)^r$$

and

$$(-1)^r\lambda_{r+1} \leqslant (\| \boldsymbol{\phi} \|/\gamma(\alpha))^{r-1} \| \mathbf{a} \|_2 \| \boldsymbol{\phi} \|/\gamma(\alpha) \| \mathbf{a} \|_2 = (\| \boldsymbol{\phi} \|/\gamma(\alpha))^r. \qquad \blacksquare$$

4.2 LEMMA. *Let* $0 < \alpha \leqslant n^{-1}$, *and let* $x_1,..., x_{n+1}$ *be points of* $[0, 1]$ *with* $\min_{k=1,\ldots,n}(x_{k+1} - x_k) \geqslant \alpha$. *Let* $\epsilon > 0$, $\psi \in H$, *and suppose that* $\min_{k=1,\ldots,n+1} (-1)^k\psi(x_k) \geqslant -\epsilon$. *Then*

$$|\psi(x_k)| \leqslant \epsilon \left(\sum_{i=1}^{n+1} (\| \boldsymbol{\phi} \|/\gamma(\alpha))^{n+i-1} - 1 \right)$$

for each k *in* $\{1,..., n+1\}$.

Proof. Compute real numbers $\lambda_1 = 1, \lambda_2,..., \lambda_{n+1}$ so that $\sum_{i=1}^{n+1} \lambda_i\boldsymbol{\phi}(x_i) = \mathbf{0}$. Noting Lemma 4.1, we have, for each k in $\{1,..., n+1\}$,

$$\begin{aligned} -\epsilon\left(\sum_{i=1}^{n+1} |\lambda_i| - |\lambda_k| \right) &\leqslant \sum_{i=1, i\neq k}^{n+1} |\lambda_i| (-1)^i \psi(x_i) \\ &= -\sum_{i=1, i\neq k}^{n+1} \lambda_i\psi(x_i) \\ &= \lambda_k\psi(x_k) \\ &= (-1)^{k-1} |\lambda_k| \psi(x_k) \\ &\leqslant \epsilon |\lambda_k|. \end{aligned}$$

It follows from this and Lemma 4.1 that

$$\begin{aligned} |\psi(x_k)| &\leqslant \epsilon \max\left(1, \sum_{i=1}^{n+1} |\lambda_i| / |\lambda_k| - 1\right) \\ &\leqslant \epsilon \max\left(1, \sum_{i=1}^{n+1} (\|\phi\|/\gamma(\alpha))^{i-1}/(\gamma(\alpha)/\|\phi\|)^n - 1\right) \\ &= \epsilon \max\left(1, \sum_{i=1}^{n+1} (\|\phi\|/\gamma(\alpha))^{n+i-1} - 1\right) \\ &= \epsilon\left(\sum_{i=1}^{n+1} (\|\phi\|/\gamma(\alpha))^{n+i-1} - 1\right). \quad \blacksquare \end{aligned}$$

4.3 LEMMA. *Let $0 < \alpha \leqslant n^{-1}$. Let $x_1, \ldots, x_n$ be points of $[0, 1]$ such that, in the case $n \geqslant 2$, $\min_{k=1,\ldots,n-1}(x_{k+1} - x_k) \geqslant \alpha$. Let $\epsilon > 0$, $\psi \in H$, and suppose that $\max_{k=1,\ldots,n} |\psi(x_k)| \leqslant \epsilon$. Then $\|\psi\| \leqslant n\gamma(\alpha)^{-1} \|\phi\|\epsilon$.*

Proof. Applying Lemma 2.1 to the sets $\{x_i\}$ $(i = 1, \ldots, n)$, we obtain

$$\epsilon \geqslant \max_{k=1,\ldots,n} |\psi(x_k)| \geqslant n^{-1}\gamma(\alpha) \|\mathbf{a}\|_2 .$$

Hence $\|\psi\| \leqslant \|\mathbf{a}\|_2 \|\phi\| \leqslant n\gamma(\alpha)^{-1} \|\phi\|\epsilon$. $\blacksquare$

We are now in a position to establish the computability of best Chebyshev approximants. Before doing so, however, we mention the following result from [4].

4.4 THEOREM. *Let F be a finite dimensional linear subspace of the normed space E over $\mathbb{R}$, and ξ an element of E with the property: $\max(\|\xi - x\|, \|\xi - x'\|) > \operatorname{dist}(\xi, F)$ whenever x, x' are distinct elements of F. Then ξ has a unique best approximant in F.*

4.5 THEOREM. *a has a best approximant b in H that is unique, in the sense that $\|a - p\| > \|a - b\| = \operatorname{dist}(a, H)$ whenever $p \in H$ and $\|p - b\| > 0$.*

Proof. In view of Theorem 4.4, it will suffice to prove that $\max(\|a - p\|, \|a - q\|) > \operatorname{dist}(a, H)$ whenever p, q belong to H and $\|p - q\| > 0$. Given such p and q, as $\|a - p\| + \|a - q\| \geqslant \|p - q\| > 0$, we lose no generality in assuming that $\|a - q\| > 0$. With δ a modulus of continuity for $a - q$ on $[0, 1]$, we choose α so that $0 < \alpha < \min(n^{-1}, 2^{-1}\delta(\|a - q\|))$, and define

$$c = \sum_{i=1}^{n+1} (\|\phi\|/\gamma(\alpha))^{n+i-1} - 1.$$

We then choose ϵ so that $0 < \epsilon < 2^{-1}\min(c^{-1} \| p - q \|, \| a - q \|)$. Either $\| a - q \| > \operatorname{dist}(a, H)$; or, as we may suppose, there exists an ϵ-alternant $(j, (x_1, ..., x_{n+1}))$ of a and q (Proposition 3.3). As $\min_{k=1,...,n+1}(-1)^{k-j}(p - q)(x_k) > -2\epsilon$ entails $\| p - q \| \leqslant c(2\epsilon) < \| p - q \|$ (Lemma 4.2), we can find k with $(-1)^{k-j}(p - q)(x_k) < -\epsilon$. For this k, we then have

$$\begin{aligned}\| a - p \| &\geqslant (-1)^{k-j}(a - p)(x_k) \\ &= (-1)^{k-j}(a - q)(x_k) + (-1)^{k-j}(q - p)(x_k) \\ &> \| a - q \| - \epsilon + \epsilon \\ &= \| a - q \|.\end{aligned}$$

Hence $\| a - p \| > \operatorname{dist}(a, H)$. ▮

When $\operatorname{dist}(a, H) > 0$, the uniqueness property of the best approximant can be strengthened (cf. [6, p. 80]).

4.6 Strong Unicity Theorem. *Let b be the best Chebyshev approximant of a in H, and suppose that $\| a - b \| > 0$. Let δ be a modulus of continuity for $a - b$ on $[0, 1]$, $\alpha = \min(n^{-1}, \delta(\| a - b \|))$ and $c = n^{-2}(\gamma(\alpha)/ \| \boldsymbol{\phi} \|)^{2n+1}$. Then*

$$\| a - p \| \geqslant \| a - b \| + c \| p - b \|$$

for each p in H.

Proof. Let $0 < \epsilon < \frac{1}{2} \| a - b \|$, construct an ϵ-alternant $(j, (x_1, ..., x_{n+1}))$ of a and b, and note that

$$\min_{k=1,...,n}(x_{k+1} - x_k) \geqslant \delta(\| a - b \|) > \alpha.$$

Compute real numbers $\lambda_1 = 1, \lambda_2, ..., \lambda_{n+1}$ so that $\sum_{i=1}^{n+1} \lambda_i \boldsymbol{\phi}(x_i) = \mathbf{0}$. We first prove

4.6.1. *Let $\psi = \sum_{i=1}^{n} a_i \phi_i$, where $\| \mathbf{a} \|_2 = 1$. Then*

$$\max_{k=1,...,n+1}(-1)^{k+j-1}\psi(x_k) \geqslant n^{-2}\gamma(\alpha)(\gamma(\alpha)/ \| \boldsymbol{\phi} \|)^{2n}.$$

We may assume that $\beta(\alpha) < | \det [\phi_k(x_i)]|$. Suppose that

$$\max_{k=1,...,n+1}(-1)^{j}\lambda_k\psi(x_k) < n^{-2}\gamma(\alpha)(\gamma(\alpha)/ \| \boldsymbol{\phi} \|)^{n}.$$

Then, choosing r in $\{1,..., n+1\}$ so that $|\psi(x_r)| > n^{-1}\gamma(\alpha)$ (Lemma 2.1), we have $(-1)^{r+j-1}\psi(x_r) \leqslant 0$: for, if $(-1)^{r+j-1}\psi(x_r) > 0$, we would have (Lemma 4.1)

$$\begin{aligned}(\gamma(\alpha)/\|\phi\|)^n(-1)^{r+j-1}\psi(x_r) &\leqslant (-1)^{r-1}\lambda_r(-1)^{r+j-1}\psi(x_r)\\ &= (-1)^j\lambda_r\psi(x_r)\\ &< n^{-2}\gamma(\alpha)(\gamma(\alpha)/\|\phi\|)^n,\end{aligned}$$

from which would follow the contradiction $0 < (-1)^{r+j-1}\psi(x_r) < n^{-2}\gamma(\alpha)$.

We now have

$$\begin{aligned}0 &= (-1)^j \sum_{k=1}^{n+1} \lambda_k\psi(x_k)\\ &< \sum_{k=1,k\neq r}^{n+1} n^{-2}\gamma(\alpha)(\gamma(\alpha)/\|\phi\|)^n + (-1)^{r-1}\lambda_r(-1)^{r+j-1}\psi(x_r)\\ &= n^{-1}\gamma(\alpha)(\gamma(\alpha)/\|\phi\|)^n - (-1)^{r-1}\lambda_r|\psi(x_r)|\\ &< n^{-1}\gamma(\alpha)(\gamma(\alpha)/\|\phi\|)^n - (\gamma(\alpha)/\|\phi\|)^{r-1} n^{-1}\gamma(\alpha)\\ &\leqslant 0.\end{aligned}$$

It follows that, in fact,

$$\begin{aligned}&n^{-2}\gamma(\alpha)(\gamma(\alpha)/\|\phi\|)^n\\ &\quad\leqslant \max_{k=1,...,n+1}(-1)^j\lambda_k\psi(x_k)\\ &\quad= \max_{k=1,...,n+1}(-1)^{k-1}\lambda_k(-1)^{k+j-1}\psi(x_k);\end{aligned}$$

whence (Lemma 4.1) $\max_{k=1,...,n+1}(-1)^{k+j-1}\psi(x_k) > 0$, and therefore

$$\begin{aligned}&n^{-2}\gamma(\alpha)(\gamma(\alpha)/\|\phi\|)^n\\ &\quad\leqslant (\|\phi\|/\gamma(\alpha))^n \max_{k=1,...,n+1}(-1)^{k+j-1}\psi(x_k).\end{aligned}$$

Thus

$$\max_{k=1,...,n+1}(-1)^{k+j-1}\psi(x_k) \geqslant n^{-2}\gamma(\alpha)(\gamma(\alpha)/\|\phi\|)^{2n},$$

and the proof of part 4.6.1 is complete.

To complete that of Theorem 4.6, let $p \in H$ and note that either $0 < \|p-b\|$ or $\|p-b\| < \epsilon$. In the former case, choose in turn $\mathbf{a} \in \mathbb{R}^n$ and $k \in \{1,..., n+1\}$ so that $p - b = \sum_{j=1}^n a_j\phi_j$ and

$$(-1)^{k+j-1}\|\mathbf{a}\|_2^{-1}(p-b)(x_k) > n^{-2}\gamma(\alpha)(\gamma(\alpha)/\|\phi\|)^{2n} - \|\phi\|\,\|p-b\|^{-1}\epsilon.$$

Then

$$\begin{aligned}
\| a - p \| &\geqslant (-1)^{k+j}(a - p)(x_k) \\
&= (-1)^{k+j}(a - b)(x_k) + (-1)^{k+j-1}(p - b)(x_k) \\
&> \| a - b \| - \epsilon + \| \mathbf{a} \|_2 (n^{-2}\gamma(\alpha)(\gamma(\alpha)/ \| \phi \|)^{2n} - \| \phi \| \, \| p - b \|^{-1}\epsilon) \\
&\geqslant \| a - b \| - \epsilon \\
&\quad + \| \phi \|^{-1} \| p - b \| (n^{-2}\gamma(\alpha)(\gamma(\alpha)/ \| \phi \|)^{2n} - \| \phi \| \, \| p - b \|^{-1}\epsilon) \\
&= \| a - b \| + c \| p - b \| - 2\epsilon.
\end{aligned}$$

In the case $\| p - b \| < \epsilon$, choosing q in H with $\| p - q \| < \min(\epsilon, c^{-1}\epsilon)$ and $\| q - b \| > 0$, we have

$$\begin{aligned}
\| a - p \| &> \| a - q \| - \epsilon \\
&\geqslant \| a - b \| + c \| q - b \| - 3\epsilon \\
&\geqslant \| a - b \| + c \| p - b \| - c \| p - q \| - 3\epsilon \\
&\geqslant \| a - b \| + c \| p - b \| - 4\epsilon.
\end{aligned}$$

Thus, in both cases, $\| a - p \| \geqslant \| a - b \| + c \| p - b \| - 4\epsilon$. As $\epsilon \in (0, \frac{1}{2} \| a - b \|)$ is arbitrary, the required result now follows. ∎

Under the conditions of Theorem 4.6, we can argue as in the classical proof of [6, p. 82, Theorem] to show that the best approximation process is locally Lipschitzian: to be exact, we have

$$\| b' - b \| \leqslant 2n^2(\| \phi \|/\gamma(\alpha))^{2n+1} \| a' - a \|$$

whenever $a' \in C[0, 1]$ and b' is the best approximant of a' in H. This estimate can be sharpened in the following manner.

Let $0 < \epsilon < \frac{1}{2} \| a - b \|$, and construct an ϵ-alternant $(j, (x_1, \ldots, x_{n+1}))$ of a and b. For each k in $\{1, \ldots, n + 1\}$ we have $(-1)^{k-j}(b' - b)(x_k) > -2 \| a' - a \| - \epsilon$ (cf. proof of [3, 6.3]). As

$$\min_{k=1,\ldots,n}(x_{k+1} - x_k) \geqslant \delta(\| a - b \|) \geqslant \alpha,$$

it follows from Lemmas 4.2 and 4.3 that

$$\| b' - b \| \leqslant n\kappa \left(\sum_{i=1}^{n+1} \kappa^{n+i-1} - 1 \right) (2 \| a' - a \| + \epsilon),$$

where $\kappa = \| \phi \|/\gamma(\alpha)$. As $\epsilon \in (0, \| a - b \|/2)$ is arbitrary, this yields the estimate

$$\| b' - b \| \leqslant 2n\kappa \left(\sum_{i=1}^{n+1} \kappa^{n+i-1} - 1 \right) \| a' - a \|.$$

To show that this is a sharper estimate than our earlier one, we need only prove that $\sum_{i=1}^{n+1} \kappa^{n+i-1} - 1 < n\kappa^{2n}$ for each positive integer n. This is a simple exercise in induction.

5. Chebyshev Approximation over Finite Sets

We now consider the case where $X = \{x_1, ..., x_{n+1}\}$, $(x_1, ..., x_{n+1})$ being a strictly increasing sequence of $n + 1$ points of $\mathbb{R}$. In this case, the classical theory [10, p. 65] tells us that there is a unique best approximant b of a in H, characterised by the property: there exists $j \in \{0, 1\}$ such that $(-1)^{k-j}$ $(a - b)(x_k) = \mathrm{dist}(a, H)$ for each k in $\{1, ..., n + 1\}$. With $b = \sum_{i=1}^{n} b_i\phi_i$ and $b_{n+1} = (-1)^j \mathrm{dist}(a, H)$, this property can be rewritten

$$\sum_{i=1}^{n} b_i\phi_i(x_k) + (-1)^k b_{n+1} = a(x_k) \qquad (k = 1, ..., n + 1). \qquad (*)$$

This gives us $n + 1$ linear equations in $b_1, ..., b_{n+1}$, and suggests the following constructive approach: solve Eqs. (*) for $b_1, ..., b_{n+1}$, and then show that $\| a - \sum_{i=1}^{n} b_i\phi_i \|_X = | b_{n+1} | = \mathrm{dist}(a, H)$.

The first of these instructions is easily carried out: it is a straightforward exercise in linear algebra to show that, in view of the Haar condition, Eqs. (*) have a unique solution for the b_i. On the other hand, we have

5.1 Theorem. *Let $X = \{x_1, ..., x_{n+1}\}$, where $(x_1, ..., x_{n+1})$ is a strictly increasing sequence of $n + 1$ points of $\mathbb{R}$. Let $(b_1, ..., b_{n+1})$ be the unique solution of the equations*

$$\sum_{i=1}^{n} b_i\phi_i(x_k) + (-1)^k b_{n+1} = a(x_k) \qquad (k = 1, ..., n + 1).$$

Then $b = \sum_{i=1}^{n} b_i\phi_i$ is a best approximant of a in H with respect to $\| \|_X$, and $\mathrm{dist}(a, H) = \| a - b \|_X = | b_{n+1} |$. *Moreover, $\| a - p \|_X > \| a - b \|_X$ whenever $p \in H$ and $\| p - b \|_X > 0$.*

Proof. It should be clear that, given p in H with $\| p - b \|_X > 0$, we need only prove that $\| a - p \|_X > | b_{n+1} |$. We first do so under the extra assumption that $| b_{n+1} | > 0$. Arguing as in Lemma 4.2, we can find $c > 0$ so that

$$\| \psi \|_X = \max_{k=1,...,n+1} | \psi(x_k) | \leq c\epsilon$$

whenever $\psi \in H$, $\epsilon > 0$ and $\min_{k=1,...,n+1}(-1)^k \psi(x_k) \geq -\epsilon$. Choose α so that $0 < \alpha < c^{-1} \| p - b \|_X$, and $j \in \{0, 1\}$ with $| b_{n+1} | = (-1)^j b_{n+1}$. As

$\min_{k=1,\ldots,n+1}(-1)^{k-j}(p-b)(x_k) > -\alpha$ entails $\| p - b \|_X \leqslant c\alpha < \| p - b \|_X$, we can find k with $(-1)^{k-j}(p-b)(x_k) < 0$. For this same k, we obtain

$$\begin{aligned} \| a - p \|_X &\geqslant (-1)^{k-j}(a-p)(x_k) \\ &= (-1)^{k-j}(a-b)(x_k) + (-1)^{k-j}(b-p)(x_k) \\ &> | b_{n+1} |, \end{aligned}$$

as we required.

In the general case, we can assume that $(p - b)(x_r) > 0$ for some r in $\{1,\ldots, n+1\}$. Then either

$$|(a-p)(x_r)| < (a-b)(x_r) = | b_{n+1} |$$

or $|(a-p)(x_r)| > (a-p)(x_r)$. In the former case, the first part of the proof immediately yields $\| a - p \|_X > | b_{n+1} |$. In the other case, we have $0 < -(a-p)(x_r)$; whence either $0 < (a-b)(x_r)$, when we again have our result from the first part of the proof; or

$$(a-p)(x_r) < (a-b)(x_r) < -(a-p)(x_r),$$

and therefore

$$| b_{n+1} | = |(a-b)(x_r)| < |(a-p)(x_r)| \leqslant \| a - p \|_X . \quad \blacksquare$$

The argument of Theorem 4.5 now yields

5.2 THEOREM. *Let $X = \{x_1, \ldots, x_{n+1}\}$, where $(x_1, \ldots, x_{n+1})$ is a strictly increasing sequence of $n+1$ points of $\mathbb{R}$. A necessary and sufficient condition that $b \in H$ be the best approximant of a in H with respect to $\| \ \|_X$ is that, for each $\epsilon > 0$, there exists j in $\{0, 1\}$ such that*

$$(-1)^{k-j}(a-b)(x_k) > \| a - b \|_X - \epsilon \qquad (k = 1,\ldots, n+1). \quad \blacksquare$$

Note that the classical proposition,

if b is the best approximant of a in H then there exists j in $\{0, 1\}$ such that $(-1)^{k-j}(a-b)(x_k) = \| a - b \|_X$ for $k = 1,\ldots, n+1$,

is essentially nonconstructive. To see this, let $x \in \mathbb{R}$ and take $X = \{0, 1\}$, $n = 1$, $\phi_1 = 1$, $a(t) = tx$ $(t = 0, 1)$. Let b be the best approximant of a in $H = \mathbb{R}\phi_1$, and suppose that there exists j in $\{0, 1\}$ such that

$$(-1)^{j+1}(a(0) - b) = (-1)^{j+2}(a(1) - b) = \| a - b \|_X .$$

Then $j = 0$ entails $x \geqslant b \geqslant 0$, while $j = 1$ entails $0 \geqslant b \geqslant x$. Thus the above classical proposition entails

$$\forall x \in \mathbb{R} \qquad (x \geqslant 0 \qquad \text{or} \qquad x \leqslant 0),$$

a statement known to be essentially nonconstructive [1, p. 26].

We conclude our paper with a link between the preceding results of this section and the case $X = [0, 1]$, to which we now return. Keeping an eye on the corresponding classical situation, we might expect that a practical method of computing the best approximant b of a in H will involve best approximations to a over suitably chosen sets of $n + 1$ distinct points of $[0, 1]$. Among such subsets of $[0, 1]$, there are certain ones whose importance for our earlier characterisation and existence theory makes them obvious candidates for our attention: namely, sets of the form $\{x_1, ..., x_{n+1}\}$ where, for some j in $\{0, 1\}$ and $\epsilon > 0$, $(j, (x_1, ..., x_{n+1}))$ is an ϵ-alternant of a and b. Our final theorem gives a measure of how far these sets live up to our expectations.

5.3 THEOREM. *Let $X = [0, 1]$, $p \in H$ and $\epsilon > 0$. Suppose that there exists an ϵ-alternant $(j, (x_1, ..., x_{n+1}))$ of a and p, and let $\underset{\sim}{p}$ be the best approximant of a in H with respect to the* sup norm *over $\{x_1, ..., x_{n+1}\}$. Then* $\max_{k=1,...,n+1} |(p - \underset{\sim}{p})(x_k)| \leqslant n\epsilon$.

Proof. Let $k \in \{1, ..., n\}$ and suppose that

$$(-1)^{k-j}((\underset{\sim}{p} - p)(x_k) + (\underset{\sim}{p} - p)(x_{k+1})) > \epsilon.$$

Then

$$\begin{aligned}
&(-1)^{k+1-j}(a - \underset{\sim}{p})(x_{k+1}) \\
&\quad = (-1)^{k+1-j}(a - p)(x_{k+1}) + (-1)^{k+1-j}(p - \underset{\sim}{p})(x_{k+1}) \\
&\quad > \|a - p\| - \epsilon + \epsilon - (-1)^{k-j}(\underset{\sim}{p} - p)(x_k) \\
&\quad \geqslant (-1)^{k-j}(a - p)(x_k) + (-1)^{k-j}(p - \underset{\sim}{p})(x_k) \\
&\quad = (-1)^{k-j}(a - \underset{\sim}{p})(x_k).
\end{aligned}$$

This is impossible, in view of Theorem 5.1. Thus

$$(-1)^{k-j}((\underset{\sim}{p} - p)(x_k) + (\underset{\sim}{p} - p)(x_{k+1})) \leqslant \epsilon. \tag{5.3.1}$$

A similar argument shows that

$$(-1)^{k-j}((\underset{\sim}{p} - p)(x_k) + (\underset{\sim}{p} - p)(x_{k+1})) \geqslant -\epsilon. \tag{5.3.2}$$

Now suppose that

$$\min_{k=1,...,n+1}(-1)^{k-j}(\underset{\sim}{p} - p)(x_k) > 0.$$

Then [2, Chap. 2, 3.3] ensures that we can construct compact sets $K_r \subset (x_r, x_{r+1})$ with $\inf\{|(\underset{\sim}{p} - p)(x)| : x \in K_r\} = 0$ for each r in $\{1, ..., n\}$. This contradicts Lemma 2.1; whence $\min_{r=1,...,n+1}(-1)^{r-j}(\underset{\sim}{p} - p)(x_r) \leqslant 0$. Given

$\alpha > 0$, we now choose r in $\{1,..., n+1\}$ with $(-1)^{r-j}(\underset{\sim}{p} - p)(x_r) < \alpha$. Using (5.3.1) and (5.3.2), we easily show that, for each s with $r+s$ in $\{1,..., n+1\}$, $(-1)^{r+s-j}(\underset{\sim}{p} - p)(x_{r+s}) \leqslant |s| \epsilon + \alpha$. As $\alpha > 0$ is arbitrary, it follows that $(-1)^{i-j}(\underset{\sim}{p} - p)(x_i) \leqslant n\epsilon$ for $i = 1,..., n+1$. On the other hand, if now $k \in \{1,..., n+1\}$ and $(-1)^{k-j}(\underset{\sim}{p} - p)(x_k) < -\epsilon$, then

$$\begin{aligned}
&(-1)^{k-j}(a - \underset{\sim}{p})(x_k) \\
&\quad = (-1)^{k-j}(a-p)(x_k) + (-1)^{k-j}(p - \underset{\sim}{p})(x_k) \\
&\quad > \| a - p \| - \epsilon + \epsilon \\
&\quad = \| a - p \|.
\end{aligned}$$

This contradicts Theorem 5.1. Thus $(-1)^{k-j}(\underset{\sim}{p} - p)(x_k) \geqslant -\epsilon$, and so $|(p - \underset{\sim}{p})(x_k)| \leqslant n\epsilon$. ∎

Under the conditions of Theorem 5.3, if also $0 < \epsilon < \frac{1}{2} \| a - p \|$ and δ is a modulus of continuity for $a - p$ on $[0, 1]$, then $\min_{k=1,...,n}(x_{k+1} - x_k) \geqslant \delta(\| a - p \|)$; so that, by Lemma 4.3,

$$\| p - \underset{\sim}{p} \| \leqslant n^2 \gamma(\delta(\| a - p \|))^{-1} \| \phi \| \epsilon.$$

In particular, if $p = b$ is the best approximant of a in H, then $\underset{\sim}{p} \to b$ as $\epsilon \to 0$. This suggests that, in order to get a practical algorithm for computing b, we look for an efficient method of constructing ϵ-alternants of a and b (without prior knowledge of b). In order to estimate the rate of convergence of $\underset{\sim}{p}$ to b we would then need to be able to compute a priori a modulus of uniform continuity for $a - b$ on $[0, 1]$. We are grateful to the referee for pointing out the following method of finding such a modulus of continuity.

We know that b is of the form $\sum_{i=1}^{n} b_i \phi_i$ with $\mathbf{b} = (b_1, ..., b_n) \in \mathbb{R}^n$. In view of the Cauchy–Schwarz inequality

$$| b(x) - b(y) | \leqslant \| \mathbf{b} \|_2 \| \phi(x) - \phi(y) \|_2 \qquad (x, y \in [0, 1]),$$

we see that a modulus of continuity for b can be obtained once we have found a bound for $\| \mathbf{b} \|_2$ independent of b. To do this, let $x_i = i/n$ $(i = 1,..., n)$ and $\Delta = \det[\phi_j(x_i)]$. Noting that $\| a - b \| = \operatorname{dist}(a, H) \leqslant \| a \|$, we see from Lemma 2.1 that

$$\begin{aligned}
\| \mathbf{b} \|_2 &\leqslant | \Delta |^{-1} n^{3/2}(n-1)! \left(\prod_{r=1}^{n} (1 + \| \phi_r \|) \right) \| b \| \\
&\leqslant | \Delta |^{-1} n^{3/2}(n-1)! \left(\prod_{r=1}^{n} (1 + \| \phi_r \|) \right) (\| a - b \| + \| a \|) \\
&\leqslant 2 \| a \| \, | \Delta |^{-1} n^{3/2}(n-1)! \prod_{r=1}^{n} (1 + \| \phi_r \|).
\end{aligned}$$

This gives us the required bound for $\| \mathbf{b} \|_2$.

Of course, there is a well-tried practical method for computing best Chebyshev approximants: the Remes algorithm. We intend to discuss that algorithm in another paper [5].

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